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Asymptotic Behaviour and Self-Similarity for the Three Dimensional Vlasov–Poisson–Fokker–Planck System

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The aim of this work is to study the asymptotic behaviour of global in time solutions of the Vlasov–Poisson–Fokker–Planck system in three dimensions. We consider both cases, with gravitational and electrostatic interaction, but disregard friction. It is proved that the distribution of particles tends for large time to the fundamental solution of the linear operator in L^1 norm, which means that the effect of the interaction potential vanishes comparatively at $t \to \infty$. In quantitative terms the result assures that the total nonlinear interaction force decays for large time with a decay rate of order t^{-3} and the potential energy behaves like $O(t^{-3/2})$. The asymptotic result is independent of the repulsive or attractive character of the interaction field. The main idea is to use the self-similarity of the fundamental solution of the linear part of the equation and the regularity of the Fokker–Planck operator in order to study the large-time distribution of particles. © 1996 Academic Press, Inc.

I. INTRODUCTION

The Fokker–Planck equation is a kinetic description for the Brownian motion of large particles in a surrounding bath. It was first used by A. D. Fokker, 1914, and M. Planck, 1917. For instance, the particles in a rarefied plasma collide with each other and their motion may be assumed to be of Brownian type. When we include the effect of the mutual interaction potential of the particles we arrive at the Vlasov–Poisson–Fokker–Planck system. This system finds application in superionic conductors, Josephson

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tunneling junctions, relaxation of dipoles, and electrons in laser light. We refer to [22] and the references therein for further models, physical interpretations, and applications of the Fokker–Planck equation. In the Appendix we give a quick review of the derivation of the model.

The purpose of the paper is to study the large-time asymptotics of the Cauchy problem for the Vlasov–Poisson–Fokker–Planck (VPFP) system without friction, which can be written as

$$\frac{\partial f}{\partial t}+(v\cdot\nabla_x)\,f+div_v(Ef)-\sigma\Delta_v f=0, \qquad \text{on } (0,T)\times\mathbb{R}^6 \tag{1.1}$$

$$E(t,x)=\theta(K*\rho(f))=\frac{\theta}{3\omega_3}\int_{\mathbb{R}^3}\frac{x-y}{|x-y|^3}\,\rho(f)(t,y)\,dy, \tag{1.2}$$

where $\sigma>0$, $\theta=\pm 1$ depending on the interaction between the particles (for electrostatic or repulsive interaction $\theta=1$, for gravitational or attractive interaction $\theta=-1$), ω_3 is the area of the two-dimensional sphere in $\mathbb{R}^3$ and $\rho(f)$ is the mass density of particles in position variable, i.e.,

$$\rho(f)(t,x)=\int_{\mathbb{R}^3} f(t,x,v)\,dv. \tag{1.3}$$

Our model is friction-less. When friction effects are included there appears an extra term $-div_v(\beta v f)$ with $\beta>0$ in the first member of Eq. (1.1), cf. [22]. We are also assuming that there is no external electric field so that the only acting field, E, is due to particle interaction. Note finally that the convolution in (1.2) is well-defined even when the mass density is a positive Radon measure (see [10]). Equations (1.1)–(1.2) must be supplemented with initial conditions. We will denote by f_o the initial state of the distribution of particles. Natural assumptions on f_o are $f_o\geqslant 0$ and $f_o\in L^1(\mathbb{R}^6)$.

There are some recent works dealing with the large-time behaviour for the VPFP system. F. Bouchut and J. Dolbeault [6] study the asymptotic behaviour of this system in the case $\beta>0$ assuming an external potential acting on the system. Using a free energy functional, considered as a Lyapunov functional, and a new compactness result, they prove that a suitable external potential produces, in some sense, a confinement of the particles and that the distribution of particles goes to the stationary solution with the same total mass.

Another related result is studied in [3] by L. L. Bonilla, J. A. Carrillo and J. Soler in which the large-time asymptotic for the system (1.1)–(1.3) in bounded domains is analyzed. In [3], reflection-type boundary conditions for f and Dirichlet boundary conditions for the potential are considered. The boundary conditions for f are specified by an operator which connect the density of the particles striking the boundary to those which have just struck it. The techniques used in this paper are based on constructing a Lyapunov functional in the same way as in [6].

Let us summarize the results about existence of solutions of the VPFP system in three dimensions. Classical solutions have been studied by H. D. Victory and B. P. O'Dwyer [25] and J. Weckler and G. Rein [23]. They require smoothness and strong decay at infinity for the data. In the more general setting of weak solutions, we can mention the works of H. D. Victory [24] and J. A. Carrillo and J. Soler [9] with initial data in L^p spaces. F. Bouchut studied in [4, 5] the regularity of the weak solutions of this system. Moreover, in [10], J. A. Carrillo and J. Soler allow for measures in Morrey spaces as initial data and prove the existence of a locally in time weak solution. Recently, A. Majda and Y. Zheng [26] and G. Majda, A. Majda and Y. Zheng [20] obtained the existence of global measure solutions in the $1-D$ case. Finally, J. A. Carrillo and J. Soler have introduced in [11, 12] the concept of functional solution and proved the global existence of a functional solution when the initial data is only a Radon measure with bounded variation.

The aim of this work is to obtain the asymptotic behaviour for our Cauchy problem, i.e., the VPFP system without friction and external potential. We will prove that the distribution of particles converges for large-time to a multiple of G, the fundamental solution of the linear equation

$$\frac{\partial f}{\partial t}+(v\cdot\nabla_x)\,f-\sigma\Delta_v f=0.$$

G is explicitly given by

$$G(t,x,v)=\frac{(3/4)^{3/2}}{(\pi\sigma)^3\,t^6}\exp\left\{-\frac{3\,|x|^2+3\,|x-tv|^2-t^2\,|v|^2}{2\sigma t^3}\right\}. \tag{1.4}$$

Our result holds for weak solutions with a certain regularity. In order to make precise such class we need some notations. Thus, we introduce the average density

$$\rho_h(f)=\int_{\mathbb{R}^3} f(t,x,-hv,v)\,dv,\qquad h\geqslant 0.$$

In the sequel $S_p(t)$ stands for

$$\max\{\|\rho_h(f)(t,\,\cdot\,)\|_{L^p(\mathbb{R}^3)},h\geqslant 0\}$$

for any $1\leqslant p\leqslant\infty$. We will write $S_p^o=S_p(0)$. The main result of this work is the following.

THEOREM 1.1. *Let (E, f) be a weak solution of* (1.1)–(1.2) *satisfying the conditions*

$$f \in C_w([0, \infty), L^1(\mathbb{R}^6)) \cap BC((0, \infty), L^1(\mathbb{R}^6)), \tag{H-1}$$

$$t^{1/2} E \in L^\infty([0, \infty), L^\infty(\mathbb{R}^3)^3) \tag{H-2}$$

$$\max\{S_{9/4}(t), 0 \leqslant t < \infty\} < \infty, \tag{H-3}$$

$$f \log f \in C([0, T]; L^1(\mathbb{R}^6)), \quad \text{for any } T > 0 \tag{H-4}$$

where C_w denotes continuity in the weak topology. Assume also that $f_o \in L^1(\mathbb{R}^6)$ is positive and has bounded initial energy, initial inertial momentum and initial entropy, i.e.,

$$(|x|^2 + |v|^2 + |\log f_o|)\, f_o \in L^1(\mathbb{R}^6) \tag{H-5}$$

and $E_o \in L^2(\mathbb{R}^3)^3$. Moreover, in the case of gravitational forces we have to assume that $f_o \in L^{9/7}(\mathbb{R}^6)$. Then,

$$\lim_{t \to \infty} \| f(t, \cdot) - MG(t, \cdot) \|_{L^1(\mathbb{R}^6)} = 0,$$

where M is the total mass of f_o.

Let us just point out that our proof also holds in the non-interaction problem (Vlasov–Fokker–Planck equation), in which case we put $\theta = 0$, hence $E = 0$. In this easier case we can drop (H-2) and (H-3) from the assumptions of Theorem 1.1., and then the same convergence result holds, implying that the solutions of the linear equation converge to a multiple of the fundamental solution. Indeed, what we show in the interaction cases, $\theta = \pm 1$, is that the linear part of the equation is asymptotically dominant.

The main ideas used in the proof are as follows: in first step we perform a re-scaling of the equation,

$$f_\lambda(t, x, v) = \lambda^{12} f(\lambda^2 t, \lambda^3 x, \lambda v) \tag{1.5}$$

and we study the equation and properties satisfied by the re-scaled quantities.

In a second step we establish suitable a priori estimates for the re-scaled systems. The regularity of the Fokker–Planck operator which was shown in [5] and in [10] will be used in order to deduce such a priori estimates, checking in particular that they hold globally in time.

Then, we take the limit $\lambda \to \infty$ in the scaling parameter using compactness results. This is equivalent in a sense to let $t \to \infty$. It is at this stage that we show that the nonlinear term of Eq. (1.1) vanishes in the limit, i.e., the effect of that term can be neglected for large time. Compactness is first proved in a weak sense (under less restrictive assumptions) and then in the L^1-norm.

Finally, we have to identify the limit. We show that it is indeed the fundamental solution G of the linear part of the equation multiplied by the total mass of the system.

Asymptotic methods based on self-similarity are a classical topic in evolution equations, cf. [1]. The present approach to asymptotic behaviour has been developed in connection with nonlinear diffusion equations, like the porous medium equation, $u_t = \Delta u^m$, $m > 1$, cf. G.I. Barenblatt and Ya. B. Zeldovich [2], S. Kamin [18], A. Friedman and S. Kamin [15], L. A. Caffarelli, J. L. Vázquez and N. I. Wolanski [7]. The four-step method outlined above was proposed in S. Kamin and J. L. Vázquez [19] and applied to the p-Laplacian evolution equation, $u_t = \Delta_p(u)$, $p > 2$. It has been applied to various other contexts, cf. M. Escobedo, J. L. Vázquez and E. Zuazua [14], V. A. Galaktionov and J. L. Vázquez [16]. A. Carpio used this type of technique in [8] to study decay rates for the velocity and velocity for the Navier–Stokes equations in two and three dimensions.

The paper is structured as follows. In Section 2 we review some of the properties of the weak solutions of this system. Section 3 is devoted to the introduction of the re-scaled systems and their properties. In Section 4 we develop the a priori estimates on the re-scaled system that we will need later. In Section 5 we discuss the weak-★ compactness of the scaled solutions and take the limit to obtain a solution of the linear system that we are able to identify as $MG(t, x, v)$. In the next section our main result on L^1-convergence is proved. Section 7 contains a brief comment on the easy extension to arbitrary space dimension $N \geqslant 3$. The final Appendix is intended to recall for the reader's convenience the physical model which leads to the Vlasov–Poisson–Fokker–Planck system and also contains a discussion of known existence results. In particular, a smallness condition on the size of the initial data is assumed along with the more standard assumptions in order to obtain solutions with the global regularity needed in this paper.

II. PROPERTIES OF WEAK SOLUTIONS

Let us note by L the operator of the linear part of the Eq. (1.1), i.e., after disregarding the term $div_v(Ef)$. The fundamental solution of L can be written as

$$\mathbf{G}(t, x, v, \xi, \nu) = G(t, x - \xi - t\nu, v - \nu) \tag{2.1}$$

where $x, v, \xi, \nu \in \mathbb{R}^3$, $t \geqslant 0$ and

$$G(t, x, v) = \frac{(3/4)^{3/2}}{(\pi\sigma)^3\, t^6} \exp\left\{ -\frac{3\,|x|^2 + 3\,|x - tv|^2 - t^2\,|v|^2}{2\sigma t^3} \right\}. \tag{2.2}$$

These formulas are obtained by Fourier transform of the operator L and applying the method of characteristics for the resulting linear first-order hyperbolic partial differential equation, cf. [22]. Integrating $\mathbf{G}$ with respect to the x variables and v variables we deduce

$$\begin{aligned} \mathscr{H}_x(t, x, \xi) &= \int_{\mathbb{R}^3} \mathbf{G}(t, x, v, \xi, \nu)\, dv \\ &= \frac{1}{((4/3)\,\pi\sigma)^{3/2}\, t^{9/2}} \exp\left\{ -\frac{|x - \xi - t\nu|^2}{(4/3)\,\sigma t^3} \right\} \end{aligned} \tag{2.3}$$

and

$$\mathscr{H}_v(t, v, \nu) = \int_{\mathbb{R}^6} \mathbf{G}(t, x, v, \xi, \nu)\, dx = \frac{1}{(4\pi\sigma t)^{3/2}} \exp\left\{ -\frac{|v - \nu|^2}{4\sigma t} \right\}. \tag{2.4}$$

From a stochastic-process point of view, $\mathbf{G}$ is called an Ornstein–Uhlenbeck diffusion process (see [22]). The properties of G will be used in an essential way in the sequel to obtain regularity (compactness in particular) for the weak solutions of the VPFP system. Such properties are not immediate from the form of the equation, which is parabolic in the v-variables but hyperbolic in x. Formulas (2.1), (2.3) and (2.4) show the interaction between the variables in the fundamental solution, which in turn can be used to show the regularity of the density in both x and v. Let us point out that the transport term $(v \cdot \nabla_x) f$ allows for diffusion in space, as we can deduce from (2.3), and not only in velocity as we could expect from the term $-\sigma \Delta_v f$.

Let us consider the term $div_v(Ef)$ in Eq. (1.1) as a second member of (1.1). If we assume that the pair (E, f) is a smooth solution of system (1.1)–(1.2), then we can write

$$\begin{aligned} f(t, x, v) &= \int_{\mathbb{R}^6} \mathbf{G}(t, x, v, \xi, \nu)\, f_o(\xi, \nu)\, d\xi\, d\nu \\ &\quad + \int_0^t \int_{\mathbb{R}^6} \nabla_\nu \mathbf{G}(t - s, x, v, \xi, \nu)\, E(s, \xi)\, f(s, \xi, \nu)\, d\xi\, d\nu\, ds. \end{aligned} \tag{2.5}$$

This formula serves as our definition of *weak* solution.

Definition 2.1. We say that (E, f) is a weak solution of (1.1)–(1.2) if E given by (1.2) is integrable with respect to the measure $f d(x, v)\, dt$, Eq. (1.1) is satisfied in the sense of distributions and (E, f) satisfies the integral Eq. (2.5).

As in [4, 10] we will consider $f = f_1 + f_2$ with f_1 and f_2 the above summands. We denote by $\rho_1(f) = \rho(f_1)$ and $\rho_2(f) = \rho(f_2)$. Using property (2.3) and performing a change of variables, we obtain the following formula for the mass density $\rho_1(f)$,

$$\rho_1(f)(t, x) = \int_{\mathbb{R}^3} \mathscr{H}_x(t, x, \xi) \int_{\mathbb{R}^3} f_o(\xi - tv, v)\, dv\, d\xi. \tag{2.6}$$

For $\rho_2(f)$ a similar representation is derived. This was pointed out firstly in [4] and this fact is very important in our development. In order to obtain an a priori estimate from (2.6) we need conditions not only on the initial mass density $\rho_o = \rho(f_o)$, but also on averages such as

$$\rho_h(f_o) = \int_{\mathbb{R}^3} f_o(\xi - hv, v)\, dv, \quad h \geqslant 0.$$

We need a control of some norm of $\rho_h(f_o)$ uniform in h. We can use norms in a L^p space as in [4] or more generally in a Morrey space as in [10].

We will denote by $KE(f, t)$, $PE(f, t)$ and $I(f, t)$ the kinetic energy, potential energy and inertial momentum respectively of the solution (E, f) of the system (1.1)–(1.2) at time $t \geqslant 0$. They are given by

$$KE(f, t) = \int_{\mathbb{R}^6} |v|^2 f(t, x, v)\, d(x, v), \tag{2.7}$$

$$PE(f, t) = -\theta \int_{\mathbb{R}^6} \Gamma(x - y)\, \rho(f)(t, x)\, \rho(f)(t, y)\, d(x, y) \tag{2.8}$$

and

$$I(f, t) = \int_{\mathbb{R}^6} |x|^2 f(t, x, v)\, d(x, v), \tag{2.9}$$

where Γ is the fundamental solution of $-\Delta$. It can be checked (see [9]) that

$$PE(f, t) = \theta\, \|E(t, \cdot)\|^2_{L^2(\mathbb{R}^3)}.$$

Throughout this work we will assume that (E, f) is a weak solution in the sense of Definition 2.1 satisfying the conditions (H-1)–(H-5) of the Introduction. There are existence results in the literature that imply this regularity. In order to make our exposition self-contained we will recall these existence results in a final Appendix. Let us remark that classical hypothesis such as the boundedness of some moments can be added if it is necessary. We consider initial data $f_o \in L^1(\mathbb{R}^6)$ positive with initial kinetic energy and inertial momentum bounded, $(|x|^2+|v|^2)\, f_o \in L^1(\mathbb{R}^6)$. The following lemma has been proved in [9].

LEMMA 2.1. *Let us consider* (E, f) *a solution satisfying the regularity* (H-1)–(H-3). *Let us also assume that* $(|x|^2+|v|^2)\, f_o \in L^1(\mathbb{R}^6)$. *Then, the following assertions hold.*

(i) *If* $f_o \in L^p(\mathbb{R}^6)$, $1 \leqslant p < \infty$, *then for any* $t>0$ *we have*

$$\|f(t, \cdot)\|_{L^p(\mathbb{R}^6)} \leqslant e^{3\beta t/p'}\, \|p_o\|_{L^p(\mathbb{R}^6)}.$$

In the case $\beta=0$ *we have a bound for the* L^p *norm of* f *uniformly in time.*

(ii) *For smooth enough solutions the kinetic energy and potential energy satisfy* (*see* [4])

$$\frac{d}{dt} KE(f, t)+\frac{d}{dt} PE(f, t)=-2\beta KE(f, t)+6\sigma\, \|f_o\|_{L^1(\mathbb{R}^6)}.$$

(iii) *We derive consequences valid for our class of solutions. In the case of electrostatic forces* $(\theta=1)$ *the potential energy is always positive. As a consequence, the kinetic energy and the inertial momentum for the system remain bounded in any time interval* $0 \leqslant t \leqslant T$, $T<\infty$. *Moreover,*

$$KE(f, t) \leqslant KE(f, 0)+\|E_o\|^2_{L^2(\mathbb{R}^3)}+6t\sigma\, \|f_o\|_{L^1(\mathbb{R}^6)}. \tag{2.10}$$

(iv) *In the case of gravitational forces* $(\theta=-1)$ *the potential energy is always negative. As a consequence, if* $f_o \in L^{9/7}(\mathbb{R}^6)$ *the kinetic energy and the inertial momentum for the system remain bounded in any time interval* $0 \leqslant t \leqslant T$, $T<\infty$. *Moreover,* $KE(f, t)$ *satisfies the following relation*

$$KE(f, t) \leqslant KE(f, 0)+6t\sigma\, \|f_o\|_{L^2(\mathbb{R}^6)}+Ce^{\beta t}\, \|f_o\|^{3/2}_{L^{9/7}(\mathbb{R}^6)}\, KE(f, t)^{1/2}. \tag{2.11}$$

(v) *Moreover, in both cases under the respective assumptions, if the kinetic energy is bounded in bounded time intervals then the inertial momentum satisfies*

$$\frac{d}{dt} I(f, t) \leqslant I(f, t)^{1/2}\, KE(f, t)^{1/2}. \tag{2.12}$$

for any $t>0$, *and as a consequence is also bounded in bounded time intervals.*

It follows from this that $KE(f, t)=O(t)$ *and* $I(f, t)=O(t^3)$ *for large* t *under the respective assumptions in the case* $\beta=0$.

The estimates on $KE(f, t)$ and $I(f, t)$ can be viewed as statements on the evolution of the mean deviation of speed, $\langle v \rangle$, and mean deviation of position, d, since

$$\langle v \rangle := \left(\int v^2 f d(x, v) \right)^{1/2} = KE(f, t)^{1/2} = O(t^{1/2}), \tag{2.13}$$

$$d := \left(\int x^2 f d(x, v) \right)^{1/2} = I(f, t)^{1/2} = O(t^{3/2}). \tag{2.14}$$

Let us remark at this point that these estimates are exact for the fundamental solution $G(t, x, v)$ which, according to the main theorem, is going to be the large-time attractor of our flow. This can be checked easily from (2.3) and (2.4). Moreover, using the Sobolev imbeddings and the characterization of PE given in the above Lemma, we can deduce

$$|PE(f, t)| \leqslant C \, \|\rho(t, \cdot)\|^2_{L^{6/5}(\mathbb{R}^6)}$$

Therefore, for G we easily obtain the expected decay estimate for the potential energy,

$$PE(G, t) = O(t^{-3/2}). \tag{2.15}$$

This was proved in [9].

III. RE-SCALING

One of the main ingredients of the proof of the large-time behaviour of the solutions of the VPFP system is the property of scale invariance (self-similarity) of the fundamental solution G of operator L, defined in the previous section. Let us review the concept of self-similar solution.

DEFINITION 3.2. $f(t, x, v)$ is a self-similar solution of the problem (1.1)–(1.2) if there exist some constants a, b, c, d such that

$$f(t, x, v) = \lambda^a f(\lambda^b t, \lambda^c x, \lambda^d v)$$

for any $\lambda > 0$ and for any $x, v \in \mathbb{R}^3$, $t \geqslant 0$.

It can be easily verified that the fundamental solution of the linear part of L is not self-similar when $\beta > 0$. However, for $\beta = 0$ the fundamental solution given by (2.1), (1.4) satisfies the following self-similarity property

$$G(t, x, v) = \lambda^{12} G(\lambda^2 t, \lambda^3 x, \lambda v). \tag{3.1}$$

Using this self-similarity we are going to define a sequence of problems which allow us to study the behaviour at infinity of the solution defined in Section 2. For any $\lambda \geqslant 1$ we set

$$f_\lambda(t, x, v) = \lambda^{12} f(\lambda^2 t, \lambda^3 x, \lambda v) \tag{3.2}$$

for any $x, v \in \mathbb{R}^3$, $t \geqslant 0$. We also set

$$E_\lambda(t, x) = \theta(K * \rho(f_\lambda)). \tag{3.3}$$

LEMMA 3.2. *The following equalities hold for any* $t > 0$, $\lambda \geqslant 1$ *and* $1 \leqslant p \leqslant \infty$.

(i)

$$\|f_\lambda(t, \cdot)\|_{L^p(\mathbb{R}^6)} = \lambda^{12/p'} \|f(\lambda^2 t, \cdot)\|_{L^p(\mathbb{R}^6)}.$$

(ii) $\rho(f_\lambda)$ *is given by* $\rho(f_\lambda)(t, x) = \lambda^9 \rho(f)(\lambda^2 t, \lambda^3 x)$. *Therefore,*

$$\|\rho(f_\lambda)(t, \cdot)\|_{L^p(\mathbb{R}^3)} = \lambda^{9/p'} \|\rho(f)(\lambda^2 t, \cdot)\|_{L^p(\mathbb{R}^3)}.$$

(iii) E_λ *is written as* $E_\lambda(t, x) = \lambda^6 E(\lambda^2 t, \lambda^3 x)$. *Then*

$$\|E_\lambda(t, \cdot)\|_{L^p(\mathbb{R}^3)^3} = \lambda^{6/p'} \|E(\lambda^2 t, \cdot)\|_{L^p(\mathbb{R}^3)^3}.$$

Proof. The first two parts are immediate. For the third part we use the definition of E_λ in (3.3) and (1.2) to have

$$E_\lambda(t, x) = \frac{\theta}{3w_3} \int_{\mathbb{R}^3} \frac{x - y}{|x - y|^3} \rho(f_\lambda)(t, y)\, dy.$$

Substituting (ii) into this formula we conclude. ∎

Since (E, f) is a weak solution of the Eq. (1.1) it satisfies (1.1) in the sense of distributions

$$\int_{Q_\infty} \left(f \frac{\partial \Phi}{\partial t} + f(v \cdot \nabla_x)\, \Phi + f(E \cdot \nabla_v)\, \Phi + f\sigma \Delta_v \Phi \right) d(t, x, v)$$
$$+ \int_{\mathbb{R}^6} \Phi(0, x, v)\, f_o(x, v)\, d(x, v) = 0, \tag{3.4}$$

where $Q_\infty = [0, \infty[\times \mathbb{R}^6$ and for any $\Phi \in C_o^\infty(Q_\infty)$. See [4, 10].

We next derive the equation satisfied by f_λ and E_λ.

LEMMA 3.3. *For each* $\lambda \geqslant 1$ *fixed, the pair* (E_λ, f_λ) *is a weak solution of the problem*

$$\frac{\partial f_\lambda}{\partial t} + (v \cdot \nabla_x)\, f_\lambda + \lambda^{-5} div_v(E_\lambda f_\lambda) - \sigma \Delta_v f_\lambda = 0, \qquad on \quad (0, T) \times \mathbb{R}^6 \tag{3.5}$$

with E_λ *given by* (3.3).

Proof. We have to prove the following equality

$$\int_{Q_T} \left(f_\lambda \frac{\partial \varphi}{\partial t} + f_\lambda (v \cdot \nabla_x)\, \varphi + \lambda^{-5} f_\lambda (E_\lambda \cdot \nabla_v)\, \varphi + f_\lambda \sigma \Delta_v \varphi \right) d(t, x, v)$$
$$+ \int_{\mathbb{R}^6} \varphi(0, x, v)\, f_{o\lambda}(x, v)\, d(x, v) = 0, \tag{3.6}$$

for any $\varphi \in C_o^\infty(Q_\infty)$, starting from (3.4). For that we define

$$\Phi(t, x, v) = \varphi(\lambda^{-2} t, \lambda^{-3} x, \lambda^{-1} v)$$

and use it as a test function in (3.4). Equation (3.6) follows after making the change of variables $t' = \lambda^2 t$, $x' = \lambda^3 x$ and $v' = \lambda^3 v$. Thus,

$$\int_{Q_\infty} f \frac{\partial \Phi}{\partial t}\, d(t, x, v) = \lambda^{14} \int_{Q_\infty} f\, (\lambda^2 t, \lambda^3 x, \lambda v) \frac{\partial \Phi}{\partial t} (\lambda^2 t, \lambda^3 x, \lambda v)\, d(t, x, v),$$

so that using the relation

$$\lambda^2 \frac{\partial \Phi}{\partial t} (\lambda^2 t, \lambda^3 x, \lambda v) = \frac{\partial}{\partial t} (\Phi(\lambda^2 t, \lambda^3 x, \lambda v)) = \frac{\partial \varphi}{\partial t} (t, x, v),$$

and the definition (3.2) of f_λ we have

$$\int_{Q_\infty} f \frac{\partial \Phi}{\partial t}\, d(t, x, v) = \int_{Q_\infty} f_\lambda \frac{\partial \varphi}{\partial t}\, d(t, x, v).$$

We proceed analogously with the other terms of (3.4). The integral version of Eq. (3.5) follows in the same manner. ▮

Concerning the moments of the functions f_λ, we have the following result.

LEMMA 3.4. *For any* $T > 0$ *and* λ *great enough (depending on* T*), there exists a constant* K*, depending on* f_o *and on* T *but not on* λ*, such that*

$$\int_{\mathbb{R}^6} (|x|^2 + |v|^2)\, f_\lambda(t, x, v)\, d(x, v) \leqslant K \tag{3.7}$$

for any $0 \leqslant t \leqslant T$.

Proof. It is immediate that

$$\begin{aligned} KE(f_\lambda, t) &= \int_{\mathbb{R}^6} |v|^2 f_\lambda(t, x, v)\, d(x, v) \\ &= \lambda^{-2} \int_{\mathbb{R}^6} |v|^2 f(\lambda^2 t, x, v)\, d(x, v) \\ &= \lambda^{-2} KE(f, \lambda^2 t). \end{aligned}$$

By Lemma 2.1.(iii) we have

$$KE(f, t) \leqslant KE(f, 0) + \|E_o\|^2_{L^2(\mathbb{R}^3)} + 6t\sigma \|f_0\|_{L^1(\mathbb{R}^6)}.$$

We conclude that $\int |v|^2 f_\lambda(t, x, v)\, d(x, v)$ is bounded independently of λ in the form

$$\int_{\mathbb{R}^6} |v|^2 f_\lambda(t, x, v)\, d(x, v) \leqslant \lambda^{-2} C_1 + C_2 t.$$

We observe that linear growth is just the growth of the kinetic energy of the fundamental solution G.

Let us now consider the inertial momentum. We have

$$\begin{aligned} I(f_\lambda, t) &= \int_{\mathbb{R}^6} |x|^2 f_\lambda(t, x, v)\, d(x, v) \\ &= \lambda^{-6} \int_{\mathbb{R}^6} |x|^2 f(\lambda^2 t, x, v)\, d(x, v) = \lambda^{-6} I(f, \lambda^2 t). \end{aligned}$$

Now we have to integrate the relation (2.12) to obtain a growth $I(f, t) = C_3 + C_4 t^3$ with constants that depend only on the initial data in the form expressed above. We get $I(f_\lambda, t) \leqslant \lambda^{-6} C_3 + C_4 t^3$. This concludes the proof of the lemma. ∎

IV. A PRIORI ESTIMATES

Uniform estimates like (3.7) are basic when passing to the limit $\lambda \to \infty$. In this section we use the equation satisfied by (E_λ, f_λ) to obtain the additional a priori estimates needed to eliminate the nonlinear term in the limit.

Thus, using (2.1) and the self-similarity of G, (3.1), we observe that f_λ satisfies the integral equation

$$\begin{aligned} f_\lambda(t, x, v) = &\int_{\mathbb{R}^6} G(t, x, v, \xi, \nu)\, f_{o\lambda}(\xi, \nu)\, d\xi\, d\nu \\ &+ \lambda^{-5} \int_0^t \int_{\mathbb{R}^6} \nabla_\nu G(t-s, x, v, \xi, \nu)\, E_\lambda(s, \xi)\, f_\lambda(s, \xi, \nu)\, d\xi\, d\nu\, ds. \end{aligned} \tag{4.1}$$

cf. (2.6). We set

$$\rho^1(f_\lambda)(t, x) = \int_{\mathbb{R}^3} \int_{\mathbb{R}^6} G(t, x, v, \xi, \nu)\, f_{o\lambda}(\xi, \nu)\, d\xi\, d\nu\, dv \tag{4.2}$$

and

$$\rho^2(f_\lambda)(t, x) = \int_{\mathbb{R}^3} \int_0^t \int_{\mathbb{R}^6} \nabla_\nu G(t-s, x, v, \xi, \nu)\, E_\lambda(s, \xi)\, f_\lambda(s, \xi, \nu)\, d\xi\, d\nu\, ds\, dv. \tag{4.3}$$

Let us recall some results about functions of the form $\rho_1(f_\lambda)$ and $\rho_2(f_\lambda)$ which can be found in [4], the last part of this lemma can be found in [10]. We will use the notation

$$S_p^i(t) = \max\{\|\rho_h^i(f)(t, \cdot)\|_{L^p(\mathbb{R}^3)}, h \geqslant 0\}$$

for any $1 \leqslant p \leqslant \infty$ and $i = 1, 2$. Recall that S_p^o stands for

$$\max\{\|\rho_h(f_o)(\cdot)\|_{L^p(\mathbb{R}^3)}, h \geqslant 0\}.$$

LEMMA 4.5. (i) *Let* $1 \leqslant p \leqslant q \leqslant \infty$. *Then,*

$$S_q^1(t) \leqslant Ct^{-9/2(1/p-1/q)}\, S_p^o, \qquad C = C(p, q, \sigma).$$

(ii) *If moreover,* $1/p - 1/9 < 1/q \leqslant 1/p$ *and* $0 \leqslant \delta < 1$, *then,*

$$S_q^2(t) \leqslant Ct^{-\gamma} \max\{t^\delta\, \|E(t, \cdot)\|_{L^\infty(\mathbb{R}^3)^3}\, S_p(t), 0 \leqslant t \leqslant T\},$$

where $C = C(p, q, \sigma)$ *and* $\gamma = \delta - 9/2(1/q - 1/p + 1/9)$.

(iii) *Let*

$$\Psi = \int_{\mathbb{R}^6} \nabla_\nu G(t, x, v, \xi, \nu)\, g(\xi, \nu)\, d\xi\, d\nu,$$

then for any $1 \leqslant p \leqslant q < \infty$ *and* $C = C(p, q, \sigma)$, *it holds*

$$\max\{\|\rho_h(\Psi)(t, \cdot)\|_{L^q(\mathbb{R}^3)}, h \geqslant 0\} \leqslant Ct^{-1/2-9/2(1/p-1/q)} \max\{\|\rho_h(g)\|_{L^p(\mathbb{R}^3)}, h \geqslant 0\}.$$

Using the elliptic relation between E and ρ, we obtain.

LEMMA 4.6. *If* $p \in L^p(\mathbb{R}^3) \cap L^q(\mathbb{R}^3)$ *and* $1 \leqslant q < 3 < p \leqslant \infty$, *then* $E = K * \rho \in L^\infty(\mathbb{R}^3)^3$ *and*

$$\|E\|_{L^\infty(\mathbb{R}^3)^3} \leqslant C\, \|\rho\|^{\alpha}_{L^q(\mathbb{R}^3)}\, \|\rho\|^{1-\alpha}_{L^p(\mathbb{R}^3)},$$

where C independent of ρ and

$$\alpha = \frac{1/p' - 2/3}{1/q - 1/p}$$

We denote by $S^i_{p,\lambda}(t)$ the quantifies $S^i_p(t)$ applied to the functions of the re-scaled systems. Some necessary a priori estimates which are independent of λ will be obtained in the following proposition. These estimates are obtained assuming that $S^o_{9/4,\lambda}$ is small enough. Then we shall see that this assumption can be dropped.

PROPOSITION 4.1. *If $\lambda^{-5} S^o_{9/4,\lambda}$ is small enough, then the following estimates hold for any t into* $(0, \infty)$:

$$S_{9/4,\lambda}(t) \leqslant 2 S^o_{9/4,\lambda}, \tag{4.4}$$

and

$$\|E_\lambda(t,\cdot)\|_{L^\infty(\mathbb{R}^3)^3} \leqslant C t^{-1/2} S^o_{9/4,\lambda}. \tag{4.5}$$

C is a constant which does not depend on λ.

Proof. It relies on estimates similar to those in the existence result of Theorem 4.1 in [10]. Set q between 9/4 and 3, then $4q/3$ takes its values on the interval between 3 and 4. Our goal is to estimate $S_{9/4,\lambda}(t)$ using (4.1) and Lemmas 4.5 and 4.6.

Combining Lemma 4.6 and the definition of $S_{p,\lambda}(t)$ we obtain

$$\|E_\lambda(t,\cdot)\|_{L^\infty(\mathbb{R}^3)^3} \leqslant C S_{q,\lambda}(t)^\alpha \, S_{4q/3,\lambda}(t)^{1-\alpha}, \tag{4.6}$$

where $\alpha = 4q/3 - 3$. Lemma 4.5 allows us to find

$$S^1_{q,\lambda}(t) \leqslant C t^{(9/2q-1)} S^o_{9/4,\lambda}$$

and

$$S^1_{4q/3,\lambda}(t) \leqslant C t^{(27/8q-2)} S^o_{9/4,\lambda}.$$

To estimate $S^2_{p,\lambda}$ which is defined by (4.3), we use Lemma 4.5(iii) with $p = q$ and we deduce

$$S^2_{p,\lambda} \leqslant C \int_0^t (t-s)^{-1/2} \left\{ S_{p,\lambda}(s) \, \|E_\lambda(s,\cdot)\|_{L^\infty(\mathbb{R}^3)^3} \right\} ds. \tag{4.7}$$

Inequality (4.7) is satisfied for all p. Combining (4.7) with Lemma 4.5 we can obtain the following estimate for $p=9/4$

$$S_{9/4,\,\lambda}(t) \leqslant S^o_{9/4,\,\lambda} + \lambda^{-5} C \int_0^t (t-s)^{-1/2} \{ S_{9/4,\,\lambda}(s) \, \|E_\lambda(s,\cdot)\|_{L^\infty(\mathbb{R}^3)^3} \} \, ds$$

and the following one for q

$$t^{(2-9/2q}S_{q,\,\lambda}(t) \leqslant S^o_{9/4,\,\lambda} + \lambda^{-5} C t^{(2-9/2q)} \times \int_0^t (t-s)^{-1/2} \{ S_{q,\,\lambda}(s) \, \|E_\lambda(s,\cdot)\|_{L^\infty(\mathbb{R}^3)^3} \} \, ds.$$

To estimate $S_{4q/3,\,\lambda}(t)$ we use Lemma 4.5(iii) to obtain

$$S_{4q/3,\,\lambda}(t) \leqslant C t^{(27/8q-2)} S^o_{9/4,\,\lambda} + \lambda^{-5} C \int_0^t (t-s)^{-1/2-9/2(1/r-3/4q)} \{ S_{r,\,\lambda}(s) \, \|E_\lambda(s,\cdot)\|_{L^\infty(\mathbb{R}^3)^3} \} \, ds$$

where $r \leqslant 4q/3$. Note that $S_{r,\,\lambda}(s)$ can be bounded using the interpolation inequality for L^p spaces. Thus, we have

$$S_{r,\,\lambda}(s) \leqslant [S_{9/4,\,\lambda}(s)]^{\alpha'} \, [S_{4q/3,\,\lambda}(s)]^{1-\alpha'}$$

where α' is given by

$$\frac{4\alpha'}{9} + \frac{3(1-\alpha')}{4q} = \frac{1}{r}.$$

Set

$$K_p(t) = \begin{cases} S_{p,\,\lambda}(t) \, t^{(2-9/2p)}, & \text{for} \quad p > \frac{9}{4} \\ S_{p,\,\lambda}(t), & \text{for} \quad p \leqslant \frac{9}{4}, \end{cases}$$

$$K_p = \max\{ K_p(t), \, 0 \leqslant t \leqslant T \}.$$

Let $B(a, b)$ be the beta function defined by

$$B(a, b) = \int_0^t (t-s)^{a-1} \, s^{b-1} \, ds.$$

It is known that if $a, b > 0$, then $B(a, b)$ is equal to

$$B(a, b) = C(a, b) \, t^{a+b-1},$$

where $C(a, b)$ is a constant which depends on a and b. Using the estimates (4.6) for E and taking into account the equalities

$$\left(\frac{27}{8q}-2\right)(1-\alpha')=\frac{9}{2r}-2,$$

and

$$1+\left(\frac{9}{2q}-2\right)\alpha+\left(\frac{27}{8q}-2\right)(1-\alpha)=\frac{1}{2},$$

we can rewrite the above estimates in the form

$$K_{9/4}(t)\leqslant S^{o}_{9/4,\,\lambda}+\lambda^{-5}CB\left(\frac{1}{2},\frac{1}{2}\right)K_{9/4}[K_q]^{\alpha}\,[K_{4q/3}]^{1-\alpha},$$

$$K_q(t)\leqslant S^{o}_{9/4,\,\lambda}+\lambda^{-5}Ct^{(2-9/2q)}B\left(\frac{1}{2},\,-\frac{3}{2}+\frac{9}{2q}\right)K_q[K_q]^{\alpha}\,[K_{4q/3}]^{1-\alpha},$$

$$K_{4q/3}(t)\leqslant S^{o}_{9/4,\,\lambda}+\lambda^{-5}Ct^{(2-27/8q)}B\left(\frac{1}{2}+\frac{27}{8q}-\frac{9}{2r},\,-\frac{3}{2}+\frac{9}{2r}\right)$$
$$\times K^{\alpha'}_{9/4}\,[K_q]^{\alpha}\,[K_{4q/3}]^{2-\alpha-\alpha'}.$$

Thus, if we assume that

$$-\frac{3}{2}+\frac{9}{2q}>0,$$

$$\frac{1}{2}+\frac{27}{8q}-\frac{9}{2r}>0,$$

and

$$-\frac{3}{2}+\frac{9}{2r}>0,$$

the above beta functions converge. For q fixed, it can be easily satisfied that for $36/13<r<3$ the above estimates hold.

Set

$$B(T)=\max\left\{B\left(\frac{1}{2},\frac{1}{2}\right),\,t^{(2-9/2q)}B\left(\frac{1}{2},\,-\frac{3}{2}+\frac{9}{2q}\right),\right.$$
$$\left.t^{(2-27/8q)}B\left(\frac{1}{2}+\frac{27}{8q}-\frac{9}{2r},\,-\frac{3}{2}+\frac{9}{2r}\right),\,0\leqslant t\leqslant T\right\}$$

and $K = \max\{K_{9/4}, K_q, K_{4q/3}\}$. Now, using the definition for the beta function, $B(a, b)$, it is straightforward to deduce that $B(T)$ does not depend on T, i.e., $B(T)$ is a constant $B(T) = B$ depending on q and r. Thus, we find

$$K \leqslant S^o_{9/4,\,\lambda} + CB\lambda^{-5}K^2. \tag{4.8}$$

Let us assume

$$S^o_{9/4,\,\lambda} \leqslant \frac{1}{4CB\lambda^{-5}}. \tag{4.9}$$

Therefore, for initial data f_o satisfying (4.9) we have

$$K \leqslant \frac{1-(1-4CB\lambda^{-5}S^o_{9/4,\,\lambda})^{1/2}}{2CB\lambda^{-5}} < \frac{1}{2CB\lambda^{-5}}. \tag{4.10}$$

Then, substituting (4.10) in (4.8) the estimate (4.4) is proved.

Using (4.6) and the definition of K_{2q} we can obtain

$$\|E_\lambda(t,\cdot)\|_{L^\infty(\mathbb{R}^3)^3} \leqslant Ct^{(9/2q-2)\,\alpha + (27/8q-2)(1-\alpha)}S^o_{9/4,\,\lambda}.$$

Since

$$1 + \left(\frac{9}{2q} - 2\right)\alpha + \left(\frac{27}{8q} - 2\right)(1-\alpha) = \frac{1}{2},$$

we finally deduce that

$$\|E_\lambda(t,\cdot)\|_{L^\infty(\mathbb{R}^3)^3} \leqslant Ct^{-1/2}S^o_{9/4,\,\lambda}. \quad \blacksquare$$

In the next lemma we will use strongly the regularity stated in Lemma 4.5.

Lemma 4.7. *Let $t > 0$ and let ε be such that $0 < \varepsilon < t$, then*

$$S_{9/4,\,\lambda}(t) \leqslant M(f, E, t)\,\lambda^{9/p'}$$

and

$$\|E_\lambda(t,\cdot)\|_{L^\infty(\mathbb{R}^3)^3} \leqslant \varepsilon^{-1/2}M(f, E, \varepsilon)\,\lambda^{9/p'},$$

for any $9/5 < p < 9/4$ and λ great enough with $M(f, E, \varepsilon)$ a constant which depends on the solution (E, f) and ε and is independent of λ.

Proof. Let us estimate $S_{9/4,\lambda}(t)$ using Lemma 4.5(ii) and (iii) with $q=9/4$ and $\delta=1/2$. Therefore, we obtain

$$S_{9/4,\lambda}(t) \leqslant Ct^{(2-9/2p)} S^{o}_{p,\lambda} + \lambda^{-5} C t^{-\gamma} \times \max\{\tau^{\delta} \, \|E_\lambda(\tau,\cdot)\|_{L^\infty(\mathbb{R}^3)^3} \, S_{p,\lambda}(\tau), 0 \leqslant \tau \leqslant t\}.$$

Using that $t^{1/2}E \in L^\infty([0,\infty), L^\infty(\mathbb{R}^3)^3$ and Lemma 3.2(iii) we have

$$\|E_\lambda(\tau,\cdot)\|_{L^\infty(\mathbb{R}^3)^3} = \lambda^6 \, \|E(\lambda^2\tau,\cdot)\|_{L^\infty(\mathbb{R}^3)^3}$$

Then, we get

$$\begin{aligned}\max\{\tau^{\delta} \, &\|E_\lambda(\tau,\cdot)\|_{L^\infty(\mathbb{R}^3)^3}, 0 \leqslant \tau \leqslant t\} \\ &\leqslant \lambda^5 \max\{\tau^{\delta} \, \|E(\tau,\cdot)\|_{L^\infty(\mathbb{R}^3)^3}, 0 \leqslant \tau \leqslant \lambda^2 t\} \\ &\leqslant \lambda^5 \, \|t^{1/2}E\|_{L^\infty([0,\infty), L^\infty(\mathbb{R}^3)^3)}.\end{aligned}$$

Combining the interpolation inequality in L^p spaces between 1 and 9/4, Lemma 3.2(ii) and the regularity (H-3) we deduce

$$\max\{S_{p,\lambda}(\tau), 0 \leqslant \tau \leqslant t\} \leqslant \max\{\|f_o\|_{L^1(\mathbb{R}^6)} \, S_{9/4}(\tau), 0 \leqslant \tau < \infty\} \, \lambda^{9/p'}.$$

With the above estimates, we conclude that

$$S_{9/4,\lambda}(t) \leqslant Ct^{(2-9/2p)} \lambda^{9/p'} S^{o}_{p} + Ct^{-\gamma} \lambda^{-5} \lambda^5 \pi(E,f) \, \lambda^{9/p'}.$$

where $\pi(E,f)$ is a constant which depends on the solution (E,f).
Therefore, the first part of the lemma is proved setting

$$M(f,E,t) = Ct^{(2-9/2p)} S^{o}_{p} + Ct^{-\gamma} \pi(E,f).$$

The proof of the second statement starts by applying Proposition 4.1 with initial data $f_\lambda(\varepsilon/2,x,v)$ instead of f_o. We must verify that

$$\lambda^{-5} S_{9/4,\lambda}\left(\frac{\varepsilon}{2}\right)$$

is small enough. With this purpose, we use the first part of the lemma to obtain

$$\lambda^{-5} S_{9/4,\lambda}\left(\frac{\varepsilon}{2}\right) \leqslant CM\left(f, E, \frac{\varepsilon}{2}\right) \lambda^{-5+9/p'}.$$

Choosing $9/5<p<9/4$, then $-5+9/p'$ and for λ great enough depending on ε we deduce that

$$\lambda^{-5} S_{9/4,\,\lambda}\left(\frac{\varepsilon}{2}\right)$$

is small enough. As a consequence, applying Proposition 4.1 with initial data $f_\lambda(\varepsilon/2, x, v)$, we have

$$\|E_\lambda(t,\cdot)\|_{L^\infty(\mathbb{R}^3)^3} \leqslant \left(t-\frac{\varepsilon}{2}\right)^{-1/2} C S_{9/4,\,\lambda}\left(\frac{\varepsilon}{2}\right).$$

Since $t>\varepsilon$ and taking into account the first part of the lemma, we finish the proof. ∎

The previous lemma leads to the following corollary.

COROLLARY 4.1. *Let ε be such that $0<\varepsilon<t$, then*

$$\int_K |E_\lambda(t,x)|\, f_\lambda(t,x,v)\, d(x,v) \leqslant M'(f,E,\varepsilon)\, \lambda^{9/p'}.$$

for any $9/5<p<9/4$, λ great enough and for any compact set K in $\mathbb{R}^6$. $M'(f,E,\varepsilon)$ is a constant which depends on the solution (E,f) and ε and is independent of λ.

Proof. We bound the above integral by the L^∞ norm of E_λ and the L^1 norm of $f_\lambda(t,x,v)$. Using Lemma 3.2(ii) we deduce that

$$\|f_\lambda(t,x,v)\|_{L^1(\mathbb{R}^6)} = \|f_o\|_{L^1(\mathbb{R}^6)}.$$

Combining this fact with the previous Lemma 4.7 we prove the result. ∎

Note that the procedure applied in Lemma 4.7 give us the maximum rate of decay of the nonlinear term in the Eq. 1.1. With this purpose we are going to improve the bound of $S_{9/4,\,\lambda}(t)$ obtained in Lemma 4.7.

COROLLARY 4.2 *Let ε_o be such that $0<6\varepsilon_o<t$, then*

$$S_{9/4,\,\lambda}(t) \leqslant M''(f,E,\varepsilon_o)$$

with M'' independent on λ.

Proof. The proof consists in the application of the ideas of Lemma 4.7. We use Lemma 4.5 and the arguments of Lemma 4.7 to obtain

$$\begin{aligned} S_{9/4,\,\lambda}(t) \leqslant{}& C\varepsilon_o^{(2-9/2p_1)} S_{p_1,\,\lambda}(t-\varepsilon_o) + C\,\|t^{1/2}E\|_{L^\infty([0,\,\infty),\,L^\infty(\mathbb{R}^3)^3)}\,\varepsilon_o^{-\gamma_1} \\ &\times \max\{S_{p_1,\,\lambda}(\tau),\ t-\varepsilon_o \leqslant \tau \leqslant t\} \end{aligned}$$

with $9/5 < p_1 < 9/4$, γ_1 given by Lemma 4.5 and the constant C does not depend on λ. Then, we use the same type of estimate to bound $S_{p_1, \lambda}(\tau)$ to deduce

$$S_{p_1, \lambda}(\tau) \leqslant C\varepsilon_o^{(9/2p_1 - 9/2p_2)} S_{p_2, \lambda}(t - 2\varepsilon_o) + C \, \|t^{1/2} E\|_{L^\infty([0, \infty), L^\infty(\mathbb{R}^3)^3)} \, \varepsilon_o^{-\gamma_2} \\ \times \max\{ S_{p_2, \lambda}(\tau'), t - 2\varepsilon_o \leqslant \tau' \leqslant \tau \}$$

for any $t - \varepsilon_o \leqslant \tau \leqslant t$ with $9/6 < p_2 < 9/5$. Moreover, choosing p_1 close enough to 9/5 we can chose p_2 close enough to 9/6. Combining the above estimates we deduce

$$S_{9/4, \lambda}(t) \leqslant C(f, E, \varepsilon_o) \, S_{p_2, \lambda}(t - 2\varepsilon_o) \\ + C(f, E, \varepsilon_o) \max\{ S_{p_2, \lambda}(\tau), t - 2\varepsilon_o \leqslant \tau \leqslant t \}.$$

Proceeding iteratively with this process we can arrive at the following estimate

$$S_{9/4, \lambda}(t) \leqslant C(f, E, \varepsilon_o) \, S_{1, \lambda}(t - 6\varepsilon_o) \\ + C(f, E, \varepsilon_o) \max\{ S_{1, \lambda}(\tau), t - 6\varepsilon_o \leqslant \tau \leqslant t \}.$$

Using the conservation of mass, we obtain the stated result. ∎

Remark. Let us estimate the decay of the nonlinear term. In Eq. (3.5) the factor of the term $div_v(E_\lambda f_\lambda)$ is λ^{-5} which implies a relative decay like $O(t^{-5/2})$. On the other hand, using the previous result and following the procedure of Corollary 4.1 we get as a conclusion

$$\int_{\mathbb{R}^6} |E(\lambda^2 t, x)| \, f(\lambda^2 t, x, v) \, d(x, v) \leqslant M''(f, E, \varepsilon_o) \, \lambda^{-6}.$$

Hence, for $t = 1$ and $\lambda = t^{1/2}$ we have proved the following additional estimate of the interaction force of a weak solution

$$\int_{\mathbb{R}^6} |E(t, x)| \, f(t, x, v) \, d(x, v) \leqslant M''(f, E, \varepsilon_o) \, t^{-3}.$$

In the next section we will use these estimates in order to pass to the limit $\lambda \to \infty$ in the re-scaled system. The crucial point is compactness.

V. WEAK-★ COMPACTNESS AND PASSAGE TO THE LIMIT

In order to obtain compactness of the sequence $\{f_\lambda\}_{\lambda \geqslant 1}$, we will use the following generalization of Ascoli–Arzela's theorem (see [17]).

THEOREM 5.2. *Assume that* I *is a topological separable space,* X *a Banach separable space and* X' *its dual space. Let us denote by* $\sigma(X', X)$ *the weak-★ topology on* X'. *Let us consider a set* $\mathscr{F} \subset X'^I$, (X'^I *is the set of functions from* I *to* X'), *with the following properties*

(i) *The set* $\{g(t), g \in \mathscr{F}\}$ *is relatively compact* $\sigma(X', X)$ *for any* $t \in I$.

(ii) *The set* $\{x' \circ g, g \in \mathscr{F}\}$ *is equicontinuous for any* $x' \in X$.

Then, for any $\{g_n\} \subset \mathscr{F}$, *there exists a subsequence* $\{g_{\sigma(n)}\}$ *and* $g_o \colon I \to (X', \sigma(X', X))$ *continuous such that* $\{x' \circ g_{\sigma(n)}\}$ *tends to* $x' \circ g_o$ *uniformly in* I *for any* $x' \in X$.

In our application we consider $\mathscr{F} = \{f_\lambda(t, \cdot), \lambda \geqslant 1\}$ as a subset of the space of measures of bounded variation $X' = \mathscr{M}(\mathbb{R}^6)$. Then, let us verify the first hypothesis of Theorem 5.2. With this aim we are going to use the following characterization of the relatively weak-★ compact subsets of $\mathscr{M}(\mathbb{R}^6)$ due to Prohorov (see [21]).

PROPOSITION 5.2. *Let* $B > 0$ *a constant and* $\mathscr{S} \subset \mathscr{M}_B(\mathbb{R}^6)$, *where* $\mathscr{M}_B(\mathbb{R}^6)$ *is the set of Radon measures* μ *such that*

$$\|\mu\|_{\mathscr{M}(\mathbb{R}^6)} = B.$$

Then, $\mathscr{S}$ *is weak-★ relatively compact in* $\mathscr{M}_B(\mathbb{R}^6)$ *if and only if for any* $\delta > 0$ *there exists a compact set* $X \subset \mathbb{R}^6$ *such that*

$$\mu(X) > B - \delta$$

for any $\mu \in \mathscr{S}$.

Hence, we have.

LEMMA 5.8. *The set* $\mathscr{F} = \{f_\lambda(t, \cdot), \lambda \geqslant 1\}$ *is relatively compact with respect to the weak-★ topology in* $\mathscr{M}(\mathbb{R}^6)$ *for any* $t \geqslant 0$.

Proof. Using Lemma 3.2(i) we have

$$\|f_\lambda(t, x, v)\|_{L^1(\mathbb{R}^6)} = \|f_o\|_{L^1(\mathbb{R}^6)}.$$

Thus, we are in the first hypothesis of Prohorov's theorem. We are reduced to prove the second condition of Proposition 5.2.

Using Lemma 3.4 we can consider a constant C depending on the initial data and on t such that

$$\int_{\mathbb{R}^6} (|x|^2 + |v|^2)\, f_\lambda(t, x, v)\, d(x, v) \leqslant C.$$

Taking any $\delta > 0$ and $R > 0$ such that $CR^{-2} < \delta$, then

$$R^2 \int_{\mathbb{R}^6/B_R} f_\lambda(t, x, v)\, d(x, v)$$

$$\leqslant \int_{\mathbb{R}^6/B_R} (|x|^2 + |v|^2)\, f_\lambda(t, x, v)\, d(x, v) \leqslant C < \delta R^2.$$

Therefore,

$$\int_{\mathbb{R}^6/B_R} f_\lambda(t, x, v)(x, v) < \delta$$

for any $\delta > 0$, we conclude the set $\mathscr{F}$ is weak-★ relatively compact in $\mathscr{M}(\mathbb{R}^6)$. ∎

Since every f_λ is positive then any adherence value in the weak-★ topology of the set $\mathscr{F}$ is a positive measure. Let us prove the equicontinuity stated in the second hypothesis of Theorem 5.2.

LEMMA 5.9. *Consider $\Phi \in C_o^2(\mathbb{R}^6)$. The set of functions defined by*

$$t \to \Psi_\lambda(t) = \int_{\mathbb{R}^6} f_\lambda(t, x, v)\, \Phi(x, v)\, d(x, v)$$

is equicontinuous in $[0, T]$. The same result holds for $\Phi \in C_o(\mathbb{R}^6)$.

Proof. The proof is similar to that one done in [9], Lemma 8. Let $R > 0$ such that $supp(\Phi) \subset B_R$. Since $\Phi \in C_o^2(\mathbb{R}^{2N})$ we can use the fact that (E_λ, f_λ) is a weak solution of the problem (3.5) to get

$$|\Psi_\lambda(t) - \Psi_\lambda(s)|$$

$$\leqslant C(R+\sigma)\, \|f_\lambda(t, \cdot)\|_{L^1(\mathbb{R}^6)}\, |t-s|$$

$$+ \lambda^{-5}\, \|\nabla_v \Phi\|_{L^\infty(\mathbb{R}^6)} \int_s^t \int_{B_R} |E_\lambda(\tau, x)|\, f_\lambda(\tau, x, v)\, d(x, v)\, d\tau,$$

where C depends only on the $L^\infty(\mathbb{R}^6)$ norms of Φ and its derivatives. Using Lemma 3.2(i) we have

$$\|f_\lambda(t, x, v)\|_{L^1(\mathbb{R}^6)} = \|f_o\|_{L^1(\mathbb{R}^6)}.$$

To estimate the second term we use Proposition 4.1 to obtain

$$\int_s^t \int_{B_R} |E_\lambda(\tau, x)|\, f_\lambda(\tau, x, v)\, d(x, v)\, d\tau$$
$$\leqslant CR\, \|t^{1/2} E_\lambda(t, \cdot)\|_{L^\infty(\mathbb{R}^3)^3}\, \|f_\lambda(t, x, v)\|_{L^1(\mathbb{R}^6)}\, |t-s|^{1/2}.$$

Proceeding in a similar way to Lemma 4.7 we have

$$\int_s^t \int_{B_R} |E_\lambda(\tau, x)|\, f_\lambda(\tau, x, v)\, d(x, v)\, d\tau$$
$$\leqslant CR\lambda^5\, \|t^{1/2} E\|_{L^\infty(0,\, \infty,\, L^\infty(\mathbb{R}^3)^3)}\, \|f_o\|_{L^1(\mathbb{R}^6)}\, |t-s|^{1/2},$$

where we have used Lemma 3.2(ii). Therefore,

$$|\Psi_\lambda(t) - \Psi_\lambda(s)|$$
$$\leqslant C(R+\sigma)\, \|f_o\|_{L^1(\mathbb{R}^6)}\, |t-s|$$
$$+ \|\nabla_v \Phi\|_{L^\infty(\mathbb{R}^6)}\, CR\, \|t^{1/2} E\|_{L^\infty(0,\, \infty,\, L^\infty(\mathbb{R}^3)^3)}\, \|f_o\|_{L^1(\mathbb{R}^6)}\, |t-s|^{1/2}.$$

Then, the set is equicontinuous. By means of an approximation argument in a compact set the proof of the last part is an easy consequence. ∎

Therefore, we can apply Theorem 5.2 to get from any sequence of $\{\lambda_n\} \to \infty$ a subsequence, which we will denote with the same index for simplicity, such that

$$f_{\lambda_n} \to g \qquad \text{in} \quad C_{w\star}([0, T], \mathcal{M}(\mathbb{R}^6)).$$

Let us identify g. In the next lemma we will see that there is a unique possible adherence value of the set $\mathcal{F} = \{f_\lambda(t, \cdot), \lambda \geqslant 1\}$ when λ goes to ∞.

Lemma 5.10 *The limit of the sequence $\{f_\lambda(t, \cdot), \lambda \geqslant 1\}$ when $\lambda \to \infty$ is $MG(t, x, v)$ where M is the total mass of the system, i.e., the L^1 norm of f_o.*

Proof. Let us consider any sequence $\{\lambda_n\} \to \infty$, we will prove that all of them have the same limit. Using Lemma 3.3, we deduce that f_{λ_n} satisfies the following equation

$$\int_{Q_T} \left(f_{\lambda_n} \frac{\partial \varphi}{\partial t} + f_{\lambda_n} (v \cdot \nabla_x)\, \varphi + \lambda_n^{-5} f_{\lambda_n} (E_{\lambda_n} \cdot \nabla_v)\, \varphi + f_{\lambda_n} \sigma \Delta_v \varphi \right) d(t, x, v)$$
$$+ \int_{\mathbb{R}^6} \varphi(0, x, v)\, f_{o\lambda_n}(x, v)\, d(x, v) = 0,$$

where $Q_T = [0, T[\times \mathbb{R}^6$ and for any $\varphi \in C_o^\infty(Q_T)$. Let us consider $\varphi \in C_o^\infty((0, T) \times \mathbb{R}^6)$ then

$$\int_{Q_T} \left(f_{\lambda_n} \frac{\partial \varphi}{\partial t} + f_{\lambda_n}(v \cdot \nabla_x)\, \varphi + \lambda_n^{-5} f_{\lambda_n}(E_{\lambda_n} \cdot \nabla_v)\, \varphi + f_{\lambda_n} \sigma \Delta_v \varphi \right) d(t, x, v) = 0.$$

Using, that f_{λ_n} converges to g in $C_{w\star}([0, T], \mathscr{M}(\mathbb{R}^6))$ we see that

$$\int_{Q_T} \left(f_{\lambda_n} \frac{\partial \varphi}{\partial t} + f_{\lambda_n}(v \cdot \nabla_x)\, \varphi + f_{\lambda_n} \sigma \Delta_v \varphi \right) d(t, x, v)$$

goes to

$$\int_{Q_T} \left(g \frac{\partial \varphi}{\partial t} + g(v \cdot \nabla_x)\, \varphi + g \sigma \Delta_v \varphi \right) d(t, x, v),$$

as $\lambda_n \to \infty$. Now, we apply Corollary 4.1 or the remark following Corollary 4.2 to deduce that

$$\int_{Q_T} \lambda_n^{-5} f_{\lambda_n}(E_{\lambda_n} \cdot \nabla_v)\, \varphi \, d(t, x, v) \leqslant M'(f, E, \varepsilon)\, \lambda_n^{-5 + 9/p'}$$

for some $\varepsilon > 0$ with λ_n great enough because $\varphi \in C_o^\infty((0, T) \times \mathbb{R}^6)$. Thus, since $9/5 < p < 9/4$ we have that

$$\int_{Q_T} \lambda_n^{-5} f_{\lambda_n}(E_{\lambda_n} \cdot \nabla_v)\, \varphi \, d(t, x, v)$$

goes to zero when n goes to infinity. Therefore, g satisfies that

$$\int_{Q_T} \left(g \frac{\partial \varphi}{\partial t} + g(v \cdot \nabla_x)\, \varphi + g \sigma \Delta_v \varphi \right) d(t, x, v) = 0$$

for any $\varphi \in C_o^\infty((0, T) \times \mathbb{R}^6)$. As a consequence, g is a weak solution of the equation

$$\frac{\partial g}{\partial t} + (v \cdot \nabla_x)\, g - \sigma \Delta_v g = 0, \qquad \text{on} \quad (0, T) \times \mathbb{R}^6. \tag{5.1}$$

Using the fact that f_{λ_n} converges to g in $C_{w\star}([0, T], \mathscr{M}(\mathbb{R}^6))$, then $f_{o\lambda_n}$ converges to $g(0, \cdot)$ in the weak-★ topology of $\mathscr{M}(\mathbb{R}^6)$. Since

$$f_{o\lambda}(x, v) = \lambda^{12} f_o(\lambda^3 x, \lambda v),$$

it is a simple matter to check that $f_{o\lambda_n}$ converges to $M\delta_o$ in the weak-★ topology of $\mathscr{M}(\mathbb{R}^6)$, where δ_o is the Dirac delta centered at zero.

Therefore, g is a weak solution of the linear Eq. (5.1) with $g \in C_{w\star}([0, T], \mathscr{M}(\mathbb{R}^6))$ and with initial condition $g(0, \cdot) = M\delta_o$. On the other hand, (5.1) have a unique solution in the sense of distributions with $g \in C([0, \infty), S'(\mathbb{R}^6))$ where $S'(\mathbb{R}^6)$ is the space of tempered distributions. Then, $g(t, x, v)$ must be $MG(t, x, v)$ because $G(t, x, v)$ is the fundamental solution of this operator. ∎

Hence, we can state the following weak convergence result, which expresses our asymptotic conclusion in a weak form.

THEOREM 5.3. *Let (E, f) be a weak solution of the system* (1.1)–(1.2) *satisfying the regularity* (H-1)–(H-3). *Let $f_o \in L^1(\mathbb{R}^6)$ with f_o positive such that the initial kinetic energy and inertial momentum are bounded, $(|x|^2 + |v|^2) f_o \in L^1(\mathbb{R}^6)$. Then, the*

$$\lim_{t \to \infty} \int_{\mathbb{R}^6} [f(t, x, v) - MG(t, x, v)]\, \Phi(t^{-3/2}x, t^{-1/2}v)\, d(x, v) = 0$$

when t goes to ∞ for any $\Phi \in C_o^\infty(\mathbb{R}^6)$.

Proof. The previous lemma ensure us that the limit of the family

$$\{f_\lambda(t, \cdot), \lambda \geqslant 1\}$$

when λ goes to ∞, is $MG(t, x, v)$. Set $t = 1$. Then the limit of

$$\int_{\mathbb{R}^6} [f_\lambda(1, x, v) - MG(1, x, v)]\, \Phi(x, v)\, d(x, v)$$

is zero when λ goes to ∞. Using the definition of f_λ in (3.2) and setting $\lambda = t^{1/2}$ we deduce that the limit of

$$\int_{\mathbb{R}^6} [t^6 f(t, t^{3/2}x, t^{1/2}x) - MG(1, x, v)]\, \Phi(x, v)\, d(x, v)$$

is zero when λ goes to ∞.

The self-similarity of G (3.1) means that

$$t^6 G(t, t^{3/2}x, t^{1/2}v) = G(1, x, v).$$

Thus, the limit of

$$\int_{\mathbb{R}^6} [t^6 f(t, t^{3/2}x, t^{1/2}v) - Mt^6 G(t, t^{3/2}x, t^{1/2}v)]\, \Phi(x, v)\, d(x, v)$$

is zero when λ goes to ∞. Using the change of variables $t' = \lambda^2 t$, $x' = \lambda^3 x$ and $v' = \lambda^3 v$, the result is proved. ∎

VI. COMPACTNESS IN L^1

In this section we obtain compactness in L^1 under some extra hypothesis on the initial data. We first recall a regularity result proved by F. Bouchut and J. Dolbeaut about the Vlasov–Poisson–Fokker–Planck system in [6] involving an additional control of the entropy.

LEMMA 6.11. *Set $E(t,x) \in L^\infty([0,T], L^\infty(\mathbb{R}^3)^3)$ with $f_o \in L^1(\mathbb{R}^6)$ positive and*

$$\int_{\mathbb{R}^6} (|v|^2 + |\log f_o|)\, f_o(x,v)\, d(x,v) < \infty.$$

If $f \in C([0,\infty), L^1(\mathbb{R}^6))$ is a solution of

$$\frac{\partial f}{\partial t} + (v \cdot \nabla_x)\, f + div_v(Ef) - \sigma \Delta_v f = 0,$$

with initial data f_o, then

(i) $f \log f \in C([0,\infty), L^1(\mathbb{R}^6))$.

(ii) $\sqrt{f} \in C([0,\infty), L^2(\mathbb{R}^6))$.

(iii) $\nabla_v \sqrt{f} \in L^2([0,T] \times \mathbb{R}^6)$ *for any* $T > 0$.

(iv) $\nabla_v f \in L^1(]0,t[\times \mathbb{R}^6)$ *for any* $T > 0$ *with*

$$\nabla_v f = 2\sqrt{f}\, \nabla_v \sqrt{f}.$$

(v) *It holds*

$$\frac{d}{dt} \int_{\mathbb{R}^6} f \log f\, d(x,v) = -4\sigma \int_{\mathbb{R}^6} |\nabla_v \sqrt{f}|^2\, d(x,v).$$

In view of this result we make in the sequel the following new assumption on the initial condition:

$$\int_{\mathbb{R}^6} f_o(x,v) \log f_o(x,v)\, d(x,v) < \infty. \tag{6.1}$$

Remark. The conditions of the previous lemma are fulfilled, for instance, under the assumptions in [4] for obtaining existence of solution. If $f_o \in L^1(\mathbb{R}^6) \cap L^\infty(\mathbb{R}^6)$ and the moments in v are bounded for all $k < m$, with $m > 6$,

$$\int_{\mathbb{R}^6} |v|^k\, f_o(x,v)\, d(x,v) < \infty$$

then, F. Bouchut proved the existence of a unique weak solution satisfying

$$f \in C([0, \infty), L^1(\mathbb{R}^6))$$

and $E(t, x) \in L^\infty([0, T], L^\infty(\mathbb{R}^3)^3)$. Thus, the previous lemma implies that

$$\int_{\mathbb{R}^6} f(t, x, v) \log f(t, x, v)\, d(x, v) \leqslant \int_{\mathbb{R}^6} f_o(x, v) \log f_o(x, v)\, d(x, v).$$

Arguing as in Lemmas 4.7 and 4.1 we obtain the following result for the re-scaled systems, under the new assumption (6.1).

LEMMA 6.12. *For any* $0 < \varepsilon < T$ *then*

(i) $\sqrt{f_\lambda} \in C([\varepsilon, \infty), L^2(\mathbb{R}^6))$ *and* $\nabla_v \sqrt{f_\lambda} \in L^2([\varepsilon, T] \times \mathbb{R}^6)$ *for any* $T > 0$ *with*

$$\nabla_v f_\lambda = 2 \sqrt{f_\lambda}\, \nabla_v \sqrt{f_\lambda}$$

and

$$\| f_\lambda(t, \cdot) \log f_\lambda(t, \cdot) \|_{L^1(\mathbb{R}^6)} \leqslant (\| f_o \log f_o \|_{L^1(\mathbb{R}^6)} + 12 \log(\lambda)\, \| f_o \|_{L^1(\mathbb{R}^6)}) + C$$

for any $t \geqslant 0$, *where* C *does not depend on* λ.

(ii) *Let us denote by* $h_\lambda(t, x, v) = \lambda^{-5} (E_\lambda \cdot \nabla_v) f_\lambda$, *then*

$$\| h_\lambda \|_{L^2([\varepsilon, T], L^1(\mathbb{R}^6))} \leqslant L$$

where L *is a constant independent of* λ, *with* λ *great enough.*

Proof. Since $E_\lambda(t, x) \in L^\infty([\varepsilon, T], L^\infty(\mathbb{R}^3)^3)$ due to the regularity (H-2), then the first part of the lemma follows easily from Lemma 6.11. Moreover,

$$\frac{d}{dt} \int_{\mathbb{R}^6} f_\lambda \log f_\lambda\, d(x, v) = -4\sigma \int_{\mathbb{R}^6} |\nabla_v \sqrt{f_\lambda}|^2\, d(x, v),$$

for any $0 < \varepsilon \leqslant t \leqslant T$. Thus, using the hypothesis on the solution, estimate (13) in [13] and Lemma 3.4, we conclude that

$$\| f_\lambda(t, \cdot) \log f_\lambda(t, \cdot) \|_{L^1(\mathbb{R}^6)} \leqslant \| f_{o\lambda} \log f_{o\lambda} \|_{L^1(\mathbb{R}^6)} + C$$

for any $t \geqslant 0$. Applying the definition of $f_{o\lambda}$ we deduce that

$$\| f_{o\lambda} \log f_{o\lambda} \|_{L^1(\mathbb{R}^6)} \leqslant (\| f_o \log f_o \|_{L^1(\mathbb{R}^6)} + 12 \log(\lambda)\, \| f_o \|_{L^1(\mathbb{R}^6)}).$$

Then, finally we show that

$$\|f_\lambda(t,\cdot)\log f_\lambda(t,\cdot)\|_{L^1(\mathbb{R}^6)} \leqslant (\|f_o \log f_o\|_{L^1(\mathbb{R}^6)} + 12\log(\lambda)\,\|f_o\|_{L^1(\mathbb{R}^6)}) + C$$

for any $t \geqslant 0$.

For the second part of the lemma we apply Lemma 4.7 to deduce

$$\|E_\lambda(t,\cdot)\|_{L^\infty(\mathbb{R}^3)^3} \leqslant M(f, E, \varepsilon)\,\lambda^{9/p'}$$

for any $9/5 < p < 9/4$, with C depending on ε and f_o with λ great enough for any $\varepsilon \leqslant t \leqslant T$.

Then, using Holder's inequality we have

$$\|h_\lambda\|_{L^2([\varepsilon,T],\,L^1(\mathbb{R}^6))} \leqslant \lambda^{-5}\,\|E_\lambda\|_{L^\infty([\varepsilon,T]\times\mathbb{R}^3)}\,\|\nabla_v f_\lambda\|_{L^2([\varepsilon,T],\,L^1(\mathbb{R}^6))}.$$

The first part of the lemma and Holder's inequality again show that

$$\|\nabla_v f_\lambda\|_{L^2([\varepsilon,T],\,L^1(\mathbb{R}^6))} \leqslant C\,\|\sqrt{f_\lambda}\|_{L^2([\varepsilon,T]\times\mathbb{R}^6)}\,\|\nabla_v\sqrt{f_\lambda}\|_{L^2([\varepsilon,T]\times\mathbb{R}^6)}.$$

Combining the above estimates and the first part of this lemma we conclude that

$$\begin{aligned}\|h_\lambda\|_{L^2([\varepsilon,T],\,L^1(\mathbb{R}^6))} \leqslant C\lambda^{-5+9/p'}\,\|f_o\|_{L^1(\mathbb{R}^6)}\,2(&\|f_o\log f_o\|_{L^1(\mathbb{R}^6)}\\ &+12\log(\lambda)\,\|f_o\|_{L^1(\mathbb{R}^6)} + C).\end{aligned}$$

Since $9/5 < p < 9/4$ we deduce that the above function of λ is bounded independently for λ for any λ great enough. Hence, the result is proved. ∎

The previous lemma allows us to apply the L^1-compactness due to F. Bouchut and J. Dolbeaut for the Vlasov–Poisson–Fokker–Planck equation in [6]. A similar result had been used by R. DiPerna and P. L. Lions in the study of Fokker–Planck–Boltzmann equations, [13].

LEMMA 6.13. *Let $\sigma, T > 0$. Let $f_o \in L^1(\mathbb{R}^6)$, $h \in L^1([0,T]\times\mathbb{R}^6)$. If $f \in C([0,T), L^1(\mathbb{R}^6))$ is a solution of*

$$\frac{\partial f}{\partial t} + (v\cdot\nabla_x)\,f - \sigma\Delta_v f = h,$$

with initial date $f_o \in F$ a bounded subset of $L^1(\mathbb{R}^6)$, with $h \in H$ a bounded subset of $L^2(0, T\colon L^1(\mathbb{R}^6))$, then for any $\eta > 0$ and ω bounded open subset of $\mathbb{R}^6$, f is compact in

$$C([\eta, T), L^1(\omega)).$$

Applying this result to sequence $\mathscr{F} = \{ f_\lambda(t, \cdot), \lambda \geqslant \lambda_o \}$ with λ_o great enough, we deduce easily the main result.

THEOREM 6.4. *Let* (E, f) *be a weak solution of the system* (1.1)–(1.2) *satisfying the regularity* (H-1)–(H-4). *Let* $f_o \in L^1(\mathbb{R}^6)$ *with* f_o *positive such that the initial kinetic energy, initial inertial momentum and initial entropy are bounded,* $(|x|^2 + |v|^2 + |\log f_o|)\, f_o \in L^1(\mathbb{R}^6)$. *Then,*

$$\lim_{t \to \infty} \| f(t, \cdot) - MG(t, \cdot) \|_{L^1(\mathbb{R}^6)} = 0.$$

Proof. Combining Lemma 6.11 and Lemma 6.12 all the assumptions of the result of compactness 6.13 are satisfied with

$$F = \{ f_\lambda(\varepsilon, \cdot), \lambda \geqslant \lambda_o(\varepsilon) \}$$

with $\varepsilon > 0$ fixed and $0 < \varepsilon < T$, and

$$H = \{ h_\lambda(t, \cdot), \lambda \geqslant \lambda_o(\varepsilon) \}.$$

Then Lemma 6.13 shows that

$$\mathscr{F} = \{ f_\lambda(t, \cdot), \lambda \geqslant \lambda_o(\varepsilon) \}$$

is compact in $C([\eta, T), L^1(\omega))$ for any $o < \varepsilon < \eta < T$ and ω bounded open subset of $\mathbb{R}^6$. As a consequence,

$$\mathscr{F} = \{ f_\lambda(t, \cdot), \lambda \geqslant \lambda_o(\varepsilon) \}$$

is compact in $C([\varepsilon, T), L^1(\omega))$ for any $0 < \varepsilon < T$ and ω bounded open subset of $\mathbb{R}^6$. Since

$$\int_{\mathbb{R}^6} (|x|^2 + |v|^2)\, f_\lambda(t, x, v)\, d(x, v) < K$$

for any $t \in [0, T]$ with K depending on f_o and on T, we find

$$\mathscr{F} = \{ f_\lambda(t, \cdot), \lambda \geqslant \lambda_o(\varepsilon) \}$$

is compact in $C([\varepsilon, T), L^1(\mathbb{R}^6))$ for any $0 < \varepsilon < T$.

Using Lemma 5.10, we conclude that the unique possible adherence value of the sequence f_λ in $C([\varepsilon, T), L^1(\mathbb{R}^6))$ is MG. Thus, arguing as in Theorem 5.3 leads to

$$\lim_{t \to \infty} \| f(t, \cdot) - MG(t, \cdot) \|_{L^1(\mathbb{R}^6)} = 0. \quad \blacksquare$$

VII. BEHAVIOUR IN N SPACE DIMENSIONS

The previous analysis is also valid for the VPFP system in any dimension $N \geqslant 3$ under comparable hypothesis. More precisely, we have the following result:

Let (E, f) *be a weak solution* (*in the sense of Definition* 2.1) *of* (1.1)–(1.2) *in* $\mathbb{R}^{2N}$ *such that*

$$f \in C_w([0, \infty), L^1(\mathbb{R}^{2N})) \cap BC((0, \infty), L^1(\mathbb{R}^{2N})), \qquad \text{(H-1)}_{\mathrm{N}}$$

$$t^{1/2}E \in L^\infty([0, \infty), L^\infty(\mathbb{R}^N)^N) \qquad \text{(H-2)}_{\mathrm{N}}$$

$$\max\{S_{3N/4}(t), 0 \leqslant t < \infty\} < \infty, \qquad \text{(H-3)}_{\mathrm{N}}$$

let us also assume that (H-4)–(H-5) *are satisfied and that the kinetic energy* $KE(f, t)$ *satisfies*

$$KE(f, t) \leqslant A + Bt \qquad \textit{for any} \quad t \geqslant 0. \tag{7.1}$$

Then, the conclusion of Theorem 1.1 *holds.*

Let us point out that condition (7.1) is fulfilled for any dimension in the electrostatic case, see [17]. In the gravitational case convenient hypotheses on f_o are needed to ensure that (7.1) holds. We recall that for $N = 3$ we have used the condition $f_o \in L^{9/7}(\mathbb{R}^6)$ precisely for that purpose in Theorem 1.1. In $N = 4$ the corresponding condition is: $f_o \in L^2(\mathbb{R}^8)$ and is small enough. Sufficient conditions for $N \geqslant 5$ involve also a control on the velocity moments of the initial data.

As for the details of the proof let us mention that for general N the correct re-scaling is given by the formula

$$f_\lambda(t, x, v) = \lambda^{4N} f(\lambda^2 t, \lambda^3 x, \lambda v),$$

which agrees with the invariance property of the fundamental solution

$$G(t, x, v) = \lambda^{4N} G(\lambda^2 t, \lambda^3 x, \lambda v).$$

Consequently, the nonlinear term in Eq. (3.5) has a factor λ^{4-3N} which goes to zero for all $N \geqslant 2$. The interaction force Ef decays in L^1-norm like $O(t^{-3(N-1)/2})$ (which gives t^{-3} when $N = 3$) and the potential energy decays like $O(t^{-3(N-2)/2})$.

Let us finally say that for $N = 2$ the method of proof fails since we are unable to control the growth of the kinetic energy (as in Lemma 2.1). In fact, the L^2 norm of $E = K * \rho$ diverges even for the density of the fundamental solution.

APPENDIX

This appendix is intended to make a quick review on the physical model which leads to the VPFP system and on the existence results related to the regularity (H-1)–(H-3). Let us make some comments on the physics of this model. From a probabilistic point of view, the Fokker–Planck equation characterizes the evolution of the probability mass density of particles in phase space. If we consider the position $x(t)$ and the velocity of the particle $v(t)$ as random variables, $f(t, x, v)$ is the probability density of these random variables which satisfies the following stochastic differential equation

$$x'' + \beta x' + \nabla \Phi_o(x) = \Gamma(t) \tag{A.1}$$

where $\beta \geqslant 0$ is the friction parameter, $\Phi_o(x)$ is the external potential acting on the physical system and $\Gamma(t)$ is a stochastic process which is Gaussian distributed with mean zero and a correlation function of the type

$$\langle \Gamma(t)\, \Gamma(t') \rangle = 2\sigma \delta_o(t - t'),$$

where δ_o is the Dirac Delta function and $\sigma > 0$, the variance of the external random force in (A.1), is called a thermal diffusion coefficient. The stochastic differential Eq. (A.1) is called a Langevin equation for the Brownian motion of the particle. It leads (see [22]) to the Fokker–Planck equation for the mass probability density, that is,

$$\frac{\partial f}{\partial t} + (v \cdot \nabla_x)\, f + div_v((E_o - \beta v)\, f) - \sigma \Delta_v f = 0, \qquad \text{on} \quad (0, T) \times \mathbb{R}^6 \tag{A.2}$$

where $E_o = -\nabla \Phi_o(t, x)$ is the external force field.

The Vlasov–Poisson–Fokker–Planck system appears when we consider a great deluge of mutually interacting particles which move in a Brownian way. Then, we must consider the self-potential induced by the particles. This potential satisfies the Poisson equation

$$-\nabla \Phi = \theta \rho(f), \tag{A.3}$$

with $\nabla \Phi$ decreasing at infinity. Thus, the force field acting on the system comes from $E(t, x) = -\nabla \Phi$. If we assume that E satisfies some suitable decreasing condition at infinity, then E is given by $E = \theta(K * \rho(f))$ where K is the gradient of the fundamental solution of the Laplacian. We have just deduced Eq. (1.2). Note that this convolution is well-defined even when the mass density is a positive Radon measure (see [10]).

Therefore, the stochastic differential Eq. (A.1) changes to

$$x'' + \beta x' + \nabla \Phi_o(x) - E(t, x) = \Gamma(t),$$

and now the Fokker–Planck equation becomes

$$\frac{\partial f}{\partial t} + (v \cdot \nabla_x) f + div_v((E + E_o - \beta v) f) - \sigma \Delta_v f = 0, \qquad \text{on} \quad (0, T) \times \mathbb{R}^6$$

which reduces to Eq. (1.1) under our assumptions $\beta = 0$ and $E_o = 0$. As a consequence, the Vlasov–Poisson–Fokker–Planck system is the Eq. (1.1) coupled with (A.3).

Let us summarize the existence results about this system. We will pay attention to the results which imply the regularity (H-1)–(H-3).

Local in time classical solutions in the three dimensional case were obtained by H. D. Victory and B. P. O'Dwyer in [25]. Later, J. Weckler and G. Rein in [23] gave sufficient conditions on the classical solutions obtained by H. D. Victory and B. P. O'Dwyer to be global in time.

J. A. Carrillo and J. Soler in [9] proved the existence of global weak solutions under the hypothesis: $f_o \in L^1(\mathbb{R}^6) \cap L^p(\mathbb{R}^6)$ with $p \geqslant (12 + 3\sqrt{5})/11$, and the initial energy and initial inertial momentum are bounded. This type of result is based on similar results for the Vlasov–Poisson system due to E. Horst and R. Hunze, [17]. Previously, H. D. Victory in [24] proved the existence of global weak solutions with the above hypothesis and $p = \infty$.

In order to prove existence of a weak solution for the system (1.1)–(1.2) (see [9]), a regularization of the system is often used by avoiding the singularity of the kernel of Poisson's equation combined with a linearization of the system. Later, a priori estimates for the sequence of approximate solutions are proved which allow to pass to the limit in the sequence of regularized problems. This general procedure was applied by J. A. Carrillo and J. Soler in [9] and [10].

F. Bouchut proved in [4] the following existence result. Given $f_o \in L^1(\mathbb{R}^6) \cap L^\infty(\mathbb{R}^6)$ with f_o positive satisfying

$$\int_{\mathbb{R}^6} |v|^k f_o(x, v)\, d(x, v) < \infty \tag{A.4}$$

for any $k < m$, with $m > 6$, there exists a unique weak solution (E, f) of the initial-value problem for system (1.1)–(1.2) satisfying the following properties

(i) $f \in C([0, \infty[, L^1(\mathbb{R}^6))$.

(ii) $t^{1/2} E \in L^\infty(0, T; L^\infty(\mathbb{R}^3)^3)$ for any $T > 0$.

(iii) $S^o_{9/4}$ is finite and $S_{9/4}(t)$ remains bounded in bounded time intervals.

(iv) The solution (E, f) satisfies (2.5).

Observe that this result does not give us the regularity needed in Theorem 1.1 since it implies (H-2)–(H-3) on bounded time intervals but not globally in time.

An existence result which ensures the necessary bounds for (H-2)–(H-3) was proved in [10]. In this work, the existence of a weak solution for initial data in spaces of measures (Morrey spaces) is obtained (and the proof can be carried out assuming that the initial data lies in L^p spaces). The result is as follows: assume that $f_o \in L^1(\mathbb{R}^6)$ with f_o positive and $S^o_{9/4} < \infty$. If

$$\max\{\|f_o\|_{L^1(\mathbb{R}^6)}, S^o_{9/4}\}$$

is *small enough*, then there exists a unique weak solution (E, f) of system (1.1)–(1.2) satisfying (2.5), the regularity (H-1)–(H-3) and moreover

(i) $f(t, \cdot)$ is positive for any $t \geqslant 0$ and its L^1 norm is preserved,

(ii) there exists a constant C depending only on σ such that for any $t \geqslant 0$

$$S_{9/4}(t) \leqslant C \max\{\|f_o\|_{L^1(\mathbb{R}^6)}, S^o_{9/4}\} \tag{A.5}$$

$$\|E(t, \cdot)\|_{L^\infty(\mathbb{R}^3)^3} \leqslant C t^{-1/2} \max\{\|f_o\|_{L^1(\mathbb{R}^6)}, S^o_{9/4}\}. \tag{A.6}$$

Let us remark that, but the smallness condition on f_o, the hypotheses on the initial data stated in the previous result are implied by those given by Bouchut, see [10]. It is an open problem worth investigating to clarify whether a smallness condition is necessary to obtain solutions in the class needed in Theorem 1.1. The answer is possibly different in the electrostatic and the gravitational cases.

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